6 Introduction to Digital Modulation

6.1. We now discuss the digital modulator-demodulator boxes shown in Figure 11. The **digital modulator** serves as the interface to the physical (analog) communication channel.

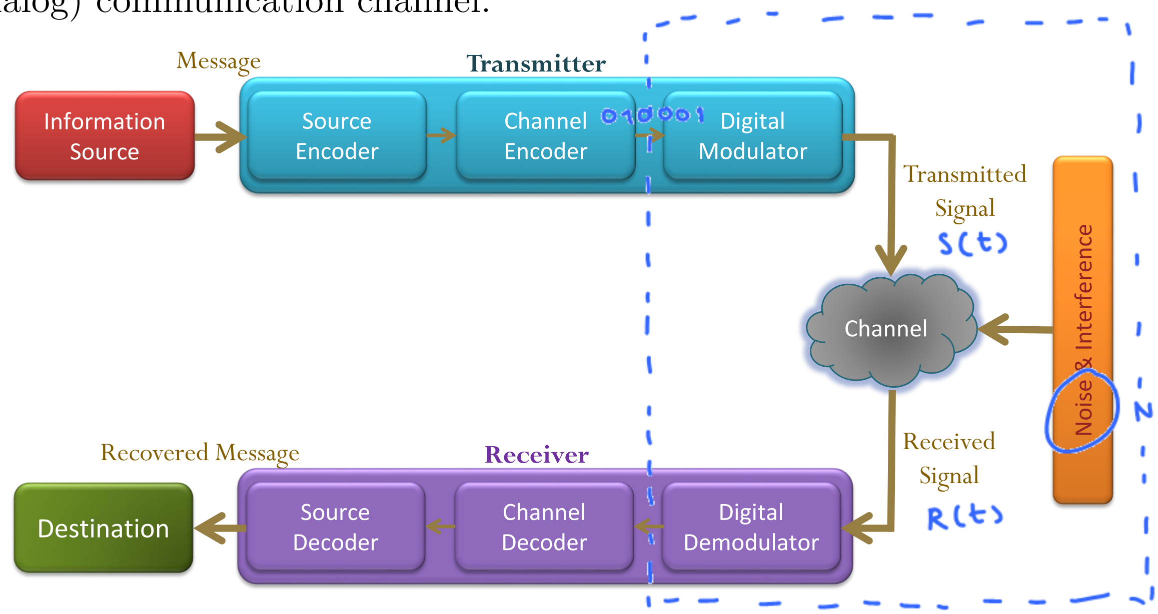
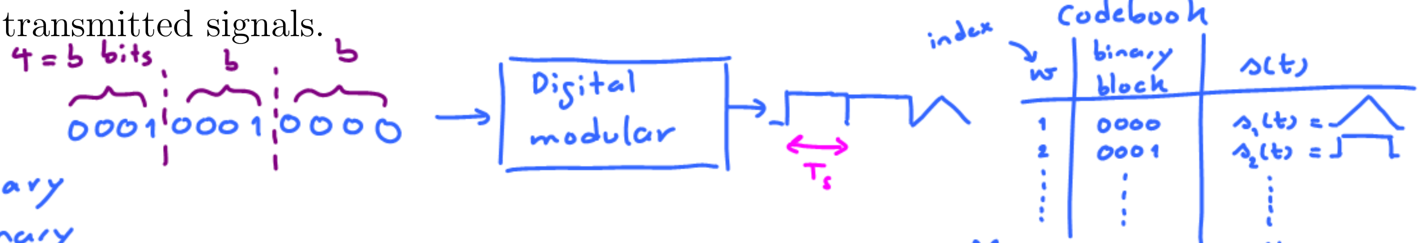


Figure 11: Basic elements of a digital communication system

The mapping between the digital sequence (which we may assume to be a binary sequence) and the (continuous-time) signal sequence to be transmitted over the channel can be either 1 memoryless or 2 with memory, resulting in memoryless modulation schemes and modulation schemes with memory.

1 **Definition 6.2.** In a **memoryless modulation** scheme, the binary sequence is parsed into blocks each of length b, and each block is mapped into one of the $s_m(t)$, $1 \le m \le 2^b$, signals regardless of the previously transmitted signals.



M=2 : binary
M=3 : ternary
M=4 : quarternary
M=5 : quinary

- This mapping from $M = 2^b$ messages to M possible signals is called ***M*-ary modulation.**
- The digital modulator may simply map the binary digit 0 into a waveform $s_1(t)$ and the binary digit 1 into a waveform $s_2(t)$. In this manner,

codebook for on-off keying

Simple ASK: ON-OFF Keying (OOK)

M = 2 $\mathcal{A} = \{0,1\}$

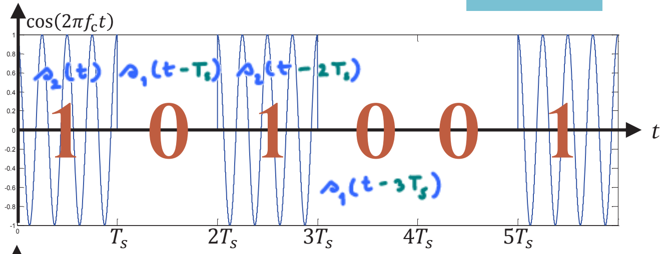
101001 → Digital Modulator → ?

$f_c = 4$ Hz
$R_s = 1$

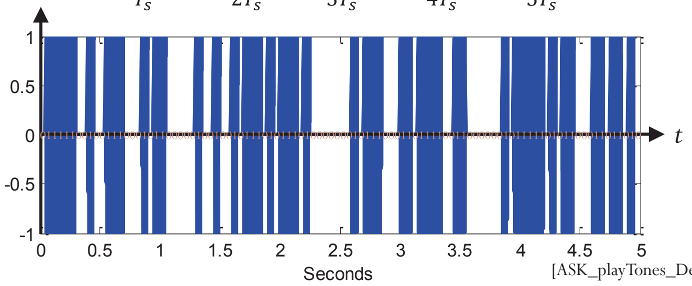
symbol rate

$T_s = \frac{1}{R_s}$

signaling interval



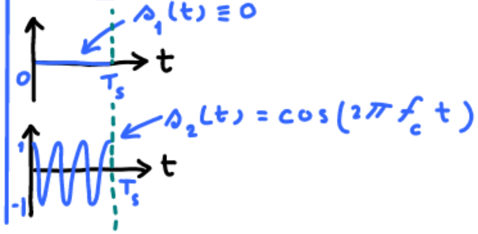
$f_c = 100$ Hz
$R_s = 20$



[ASK_playTones_Demo.m]

Codebook

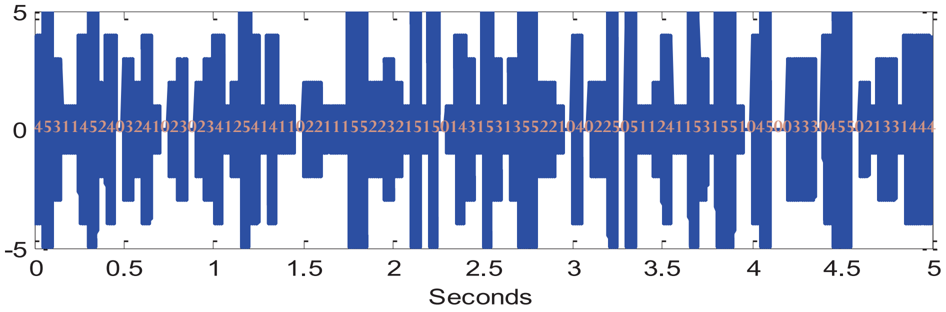
input b	index w	output
0	1	$s_1(t) \equiv 0$
1	2	$s_2(t) = \cos(2\pi f_c t)$



$$s(t) = \sum_{k=0}^{\infty} s_{w_k}(t - kT_s)$$

ASK: Higher Order Modulation

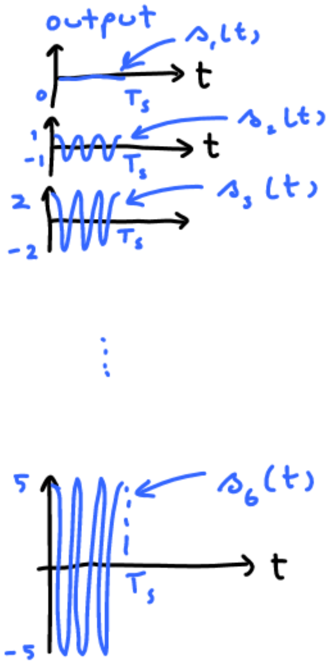
M = 6 $\mathcal{A} = \{0,1,2,3,4,5\}$



[ASK_playTones_Demo.m]

Codebook

input	index	output
'a'	1	$s_1(t)$
'b'	2	$s_2(t)$
'c'	3	$s_3(t)$
'd'	4	⋮
'e'	5	
'f'	6	$s_6(t)$



each bit from the channel encoder is transmitted separately.
We call this **binary modulation**.

- The waveforms $s_m(t)$ used to transmit information over the communication channel can be, in general, of any form. However, usually these waveforms are bandpass signals which may differ in amplitude or phase or frequency, or some combination of two or more signal parameters.

2 **Definition 6.3.** In a modulation scheme with memory, the mapping is from the set of the current b bits and the past $(L-1)b$ bits to the set of possible $M = 2^b$ messages.

- Modulation systems with memory are effectively represented by Markov chains.
- The transmitted signal depends on the current b bits as well as the most recent $L-1$ blocks of b bits.
- This defines a finite-state machine with $2^{(L-l)b}$ states.
- The mapping that defines the modulation scheme can be viewed as a mapping from the current state and the current input of the modulator to the set of output signals resulting in a new state of the modulator.
- Parameter L is called the **constraint length** of modulation.
- The case of $L = 1$ corresponds to a memoryless modulation scheme.

Definition 6.4. We assume that these signals (selected from the signal collection $\{s_1(t), s_2(t), \ldots, s_M(t)\}$ are transmitted at every T_s seconds.

- T_s is called the **signaling interval**.
- This means that in each second

$$R_s = \frac{1}{T_s}$$

symbol transmission rate

baud rate

symbols are transmitted.
Parameter R_s is called the **signaling rate** or **symbol rate**.

Definition 6.5. The energy content of a signal $s_m(t)$ is denoted by $\mathcal{E}_m$. It can be calculated from

$$E_m = \int_{-\infty}^{\infty} |s_m(t)|^2 dt.$$

The energy of a waveform $g(t)$ is given by $E_g = \int_{-\infty}^{\infty} |g(t)|^2 dt$

magnitude

(The power of a waveform $g(t)$ is given by $\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |g(t)|^2 dt$.)

1 symbol $\equiv \log_2 M$ bits

6.6. The **average signal energy** (per symbol) for the M-ary modulation in Definition 6.2 is given by

$$E_s = \sum_{m=1}^{M} p_m E_m$$

Average energy per bit

$$E_b = \frac{E_s}{\log_2 M}$$

where p_m indicates the probability of the mth signal (message probability).

Average power (sent by the Tx)

$$P = \frac{E_s}{T_s} = R_s E_s$$

- For equiprobable signals,

$$p_m = \frac{1}{M} \qquad E_s = \sum_{m=1}^{M} \frac{1}{M} E_m = \frac{1}{M}\sum_{m=1}^{M} E_m$$

- If all signals have the same energy, then
  - $E_m \equiv E$ for some E and
  - $E_s = E$.

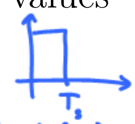
Example 6.7. In (the digital version of) **Pulse Amplitude Modulation (PAM)**, the signal waveforms are of the form

$$s_m(t) = A_m p(t), \quad 1 \le m \le M \tag{32}$$

where $p(t)$ is a (common) pulse and $\mathcal{A} = \{A_m, 1 \le m \le M\}$ denotes the set of M possible amplitudes.

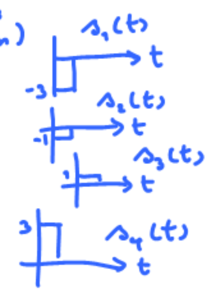
- For example, the signal "amplitudes" A_m may take the discrete values

$$A_m = 2m - 1 - M, \quad m = 1, 2, \ldots, M$$



i.e., the "amplitudes" are $\pm 1, \pm 3, \pm 5, \ldots, \pm (M-1)$.

	Amplitude (A_m)
00	-3
01	-1
11	1
10	3



- The shape of $p(t)$ influences the spectrum of the transmitted signal.
- The energy in signal $s_m(t)$ is given by

$$E_m = \int_{-\infty}^{\infty} |s_m(t)|^2 dt = \int_{-\infty}^{\infty} A_m^2 p^2(t)\, dt = A_m^2 \underbrace{\int_{-\infty}^{\infty} p^2(t) dt}_{E_p} = A_m^2 E_p$$

- For equiprobable signals,

$$E_s = \sum_{m=1}^{M} \overset{1/M}{p_m} E_m = \frac{1}{M}\sum_{m=1}^{M} E_m = \frac{1}{M}\sum_{m=1}^{M} A_m^2 E_p = \frac{E_p}{M}\sum_{m=1}^{M} A_m^2$$

Ex. $M=2$ $\quad \mathcal{A} = \{-1, 1\}$ $\quad E_s = \frac{E_p}{\cancel{2}}\left(\cancel{1^2 + (-1)^2}\right) = E_p$

Ex. $M=4$ $\quad \mathcal{A} = \{-3, -1, 1, 3\}$ $\quad E_s = \frac{E_p}{4}\left((-3)^2 + (-1)^2 + 1^2 + 3^2\right) = 5E_p$

Ex. For general M, $\quad E_s = \frac{E_p}{M}\left((-(M-1))^2 + \cdots + (-1)^2 + (1)^2 + \cdots + (M-1)^2\right) \overset{HW}{=} ?$

- Suppose $M = 2$ (binary modulation) and $s_1(t) = -s_2(t)$. The two signals have the same energy and a cross-correlation coefficient of -1. Such signals are called antipodal. This case is sometimes called **binary antipodal signaling**.

Example 6.8. In **Amplitude-Shift Keying** (ASK), the (common) pulse $p(t)$ in (32) for PAM is replaced by

$$p(t) = g(t)\cos(2\pi f_c t).$$

where f_c is the carrier frequency.

- Note that $E_p = \frac{E_g}{2}$.

6.9. The mapping or assignment of b (encoded) bits to the $M = 2^b$ possible signals may be done in a number of ways. The preferred assignment is one in which the adjacent signal amplitudes differ by one binary digit. This mapping is called **Gray coding**.

- It is important in the demodulation of the signal because the most likely errors caused by (additive white gaussian) noise involve the erroneous selection of an adjacent amplitude to the transmitted signal amplitude. In such a case, only a single bit error occurs in the b-bit sequence.

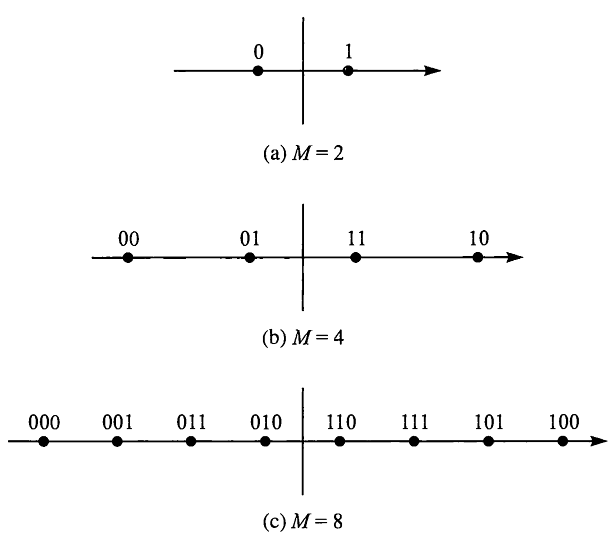


Figure 12: Gray coding for PAM signaling

6.10. In PAM (and ASK), we use just one pulse (sinusoidal pulse in the case of ASK) and modify the amplitude of the pulse to create many waveforms $s_1(t), s_2(t), \ldots s_M(t)$ that we can use to transmit different block of bits. Next, we would like to study the case where multiple shapes are used.

Example 6.11. For (baseband) binary (digital) modulation, we may use the two waveforms $s_1(t)$ and $s_2(t)$ shown in Figure 13.

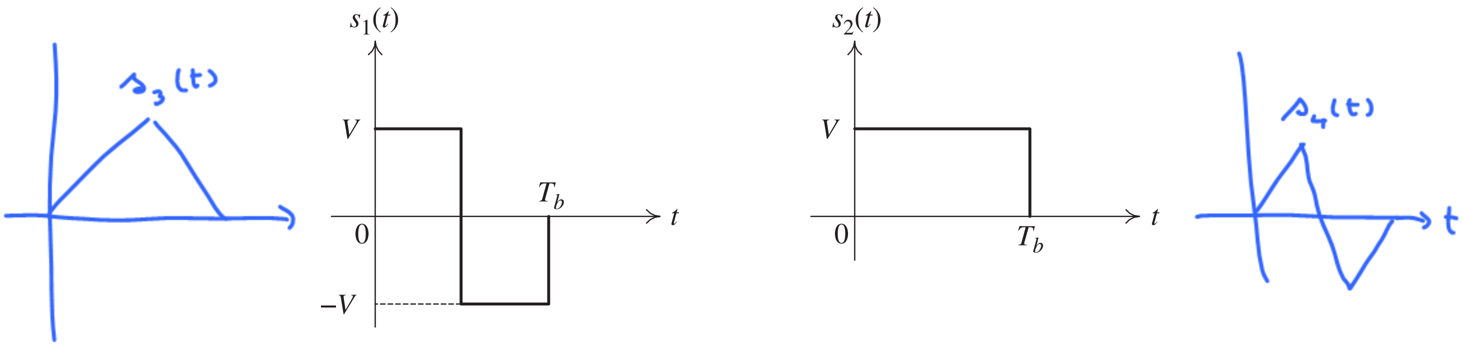


Figure 13: Signal set for Example 6.11.

Definition 6.12. The collection of all waveforms $s_1(t), s_2(t), \ldots, s_M(t)$ used in a particular digital modulation is called its **signal set**.

6.13. It is difficult to visualize, find relationship between, work with, or perform analysis directly on waveforms. For example, when we have many waveforms in the signal set, it is difficult to tell (by looking at their plots) how easy it is for them to get corrupted by the noise process; that is, how easy it is for one waveform to be interpreted as being another waveform at the demodulator.

In the next sections, we will study how to represent waveforms in the signal set as "equivalent" vectors (or points) in the signal space similar to what we saw in Figure 12. Representing waveforms as points allows us to look at them as a collection effectively.

6.14. A signal space is a vector space. So, we will first provide a review of some concepts related to vector spaces.

6.1 Vector Space and Inner Product Space in $\mathbb{C}^n$ (and $\mathbb{R}^n$)

In linear algebra, an inner product space is a vector space[15] with an additional structure called an inner product.

Definition 6.15. The **inner product** of two (potentially complex-valued) n-dimensional vectors $\mathbf{u}$ and $\mathbf{v}$ is defined as

$\dagger$ $\vec{u} = \begin{pmatrix}5\\3\end{pmatrix}$ $\vec{v} = \begin{pmatrix}1\\2\end{pmatrix}$ $\vec{w} = \begin{pmatrix}j\\-j\end{pmatrix}$

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{v}^H \mathbf{u}$$

$= \vec{v}^\dagger \vec{u}$

$\langle \vec{u}, \vec{v} \rangle = (1 \quad 2)\begin{pmatrix}5\\3\end{pmatrix} = 5 + 6 = 11$

where $(\cdot)^H$ denotes the **Hermitian transpose** operator which performs transposing operation and then conjugation.

$\langle \vec{u}, \vec{w} \rangle = (-j \quad +j)\begin{pmatrix}5\\3\end{pmatrix} = -5j + 3j$

$= -2j$

6.16. Some properties of the inner product

- $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle^*$

  $z + z^* = 2\,\mathrm{Re}\{z\} = 2\,\mathrm{Re}\{z^*\}$
- $\langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{u} \rangle = 2\,\mathrm{Re}\left\{\langle \mathbf{u}, \mathbf{v} \rangle\right\} = 2\,\mathrm{Re}\left\{\langle \mathbf{v}, \mathbf{u} \rangle\right\}$.

Definition 6.17. Two vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

More generally, a set of N vectors $\mathbf{v}^{(k)}$, $1 \le k \le N$, are **orthogonal** if $\left\langle \mathbf{v}^{(i)}, \mathbf{v}^{(j)} \right\rangle = 0$ for all $1 \le i, j \le N$, and $i \ne j$.

Definition 6.18. The **norm** of a vector $\mathbf{v}$ is denoted by $\|\mathbf{v}\|$ and is defined as

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$

which in the n-dimensional Euclidean space is simply the **length** of the vector.

Definition 6.19. A collection of vectors is said to be **orthonormal** if the vectors are orthogonal and each vector has a unit norm.

6.20. Given two vectors $\mathbf{u}$ and $\mathbf{v}$, we can **decompose** $\mathbf{v}$ into a sum of two vectors, one a multiple of $\mathbf{u}$ and the other orthogonal to $\mathbf{u}$.

(a) $\mathrm{proj}_{\mathbf{u}}(\mathbf{v}) = \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle}\mathbf{u}$ is the orthogonal projection of $\mathbf{v}$ onto $\mathbf{u}$.

Clearly, this is a multiple of $\vec{u}$. So, it is on the same "line" as $\vec{u}$.

(b) $\mathbf{v} - \mathrm{proj}_{\mathbf{u}}(\mathbf{v})$ is the component of $\mathbf{v}$ orthogonal to $\mathbf{u}$.

[15] Recall that a vector space is a mathematical structure formed by a collection of elements called vectors, which may be added together and multiplied ("scaled") by numbers, called scalars in this context.

When $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$ are real-valued $\underbrace{\text{n-D}}_{\text{n-dimensional}}$ vectors

$\langle \cdot , \cdot \rangle$ is the same as dot product that you may have seen in elementary linear algebra class.

$$\langle \vec{u}, \vec{v} \rangle = \vec{v}^H \vec{u} = \vec{v}^T \vec{u} = (v_1 \; v_2 \; \cdots \; v_n) \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = \sum_{k=1}^{n} u_k v_k = \vec{u} \cdot \vec{v}$$

$$\|\vec{u}\|^2 \equiv \langle \vec{u}, \vec{u} \rangle = \sum_{k=1}^{n} u_k u_k = \sum_{k=1}^{n} u_k^2$$

When $\vec{u}$ and $\vec{v}$ are complex-valued n-D vectors,

we need to be careful with conjugation:

$$\langle \vec{u}, \vec{v} \rangle = \vec{v}^H \vec{u} = \left(\vec{v}^T\right)^* \vec{u} = (v_1^* \; v_2^* \; \cdots \; v_n^*) \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = \sum_{k=1}^{n} u_k v_k^*$$

$$\|\vec{u}\|^2 = \langle \vec{u}, \vec{u} \rangle = \sum_{k=1}^{n} u_k u_k^* = \sum_{k=1}^{n} |u_k|^2 = \sum_{k=1}^{n} \left((\mathrm{Re}\{u_k\})^2 + (\mathrm{Im}\{u_k\})^2 \right)$$

complex conjugation of complex number

If $z = \mathrm{Re}\{z\} + j\,\mathrm{Im}\{z\}$, then $z^* = \mathrm{Re}\{z\} - j\,\mathrm{Im}\{z\}$.

Ex. $z = 3 + 4j$

$z^* = 3 - 4j$

For complex number $z = \mathrm{Re}\{z\} + j\,\mathrm{Im}\{z\}$,

$$z^* z = z z^* = |z|^2 = (\mathrm{Re}\{z\})^2 + (\mathrm{Im}\{z\})^2$$

$|z| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$

$$|z| = \sqrt{(\mathrm{Re}\{z\})^2 + \mathrm{Im}\{z\}^2}$$

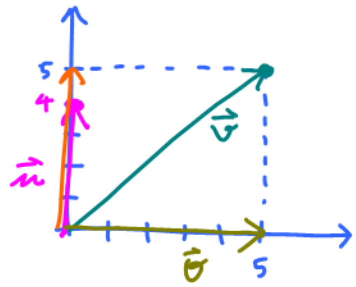
magnitude (in the complex plane)
modulus
complex norm
absolute value

Example 6.21. Let $\mathbf{v} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$ and $\mathbf{u} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$.

$\langle \vec{v}, \vec{u} \rangle = 5\times 0 + 5 \times 4 = 20$

$\langle \vec{v}, \vec{v} \rangle = 5^2 + 5^2 = 50$ $\quad \|\vec{v}\| = \sqrt{50} = 5\sqrt{2}$

$\langle \vec{u}, \vec{u} \rangle = 0^2 + 4^2 = 16$ $\quad \|\vec{u}\| = 4$



$$\vec{p} = \mathrm{proj}_{\vec{u}} \vec{v} = \frac{\langle \vec{v}, \vec{u} \rangle}{\langle \vec{u}, \vec{u} \rangle} \vec{u} = \frac{\cancel{20}^{\,5}}{\cancel{16}_{\,4}} \begin{pmatrix} 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$$

$$\vec{o} = \vec{v} - \mathrm{proj}_{\vec{u}} \vec{v} = \begin{pmatrix} 5 \\ 5 \end{pmatrix} - \begin{pmatrix} 0 \\ 5 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$$

$\vec{v} = \vec{p} + \vec{o}$

$\langle \vec{o}, \vec{u} \rangle = 0\times 5 + 4 \times 0 = 0$

$\vec{o} \perp \vec{u}$

6.22. Suppose $\mathbf{e}$ is a unit vector; that is $\|\mathbf{e}\| = 1$.

$$\mathrm{proj}_{\mathbf{e}}(\mathbf{v}) = \frac{\langle \vec{v}, \vec{e} \rangle}{\langle \vec{e}, \vec{e} \rangle} \vec{e} = \langle \vec{v}, \vec{e} \rangle \vec{e}$$

$\langle \vec{e}, \vec{e} \rangle = \|\vec{e}\|^2 = 1^2 = 1$

$\langle \vec{o}, \vec{p} \rangle = 0 \times 5 + 5 \times 0 = 0$

$\vec{o} \perp \vec{p}$

Definition 6.23. When we have a list of vectors, we use **superscripts** in parentheses as indices of vectors.

Example 6.24. Here is a list of four vectors:

$$\mathbf{v}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}^{(2)} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \mathbf{v}^{(3)} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \text{ and } \mathbf{v}^{(4)} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}.$$

As usual, subscripts represent element indices inside individual vectors. So, for the second vector, we have $v_1^{(2)} = 1$, $v_2^{(2)} = -1$, and $v_3^{(2)} = 0$.

6.25. Any vector in a vector space may also be represented as a linear combination of orthogonal unit vectors or an **orthonormal basis** $\{\mathbf{e}^{(i)}, 1 \le i \le N\}$ (for that vector space), i.e.,

$$\mathbf{v} = \sum_{i=1}^{N} \mathrm{proj}_{\mathbf{e}^{(i)}}(\mathbf{v}) = \sum_{i=1}^{N} c_i \mathbf{e}^{(i)}$$

linear combination of vectors in the orthonormal basis

where, by definition, a unit vector has length unity and c_i is the projection of the vector $\mathbf{v}$ onto the unit vector $\mathbf{e}^{(i)}$, i.e.,

$$c_i = \left\langle \mathbf{v}, \mathbf{e}^{(i)} \right\rangle.$$

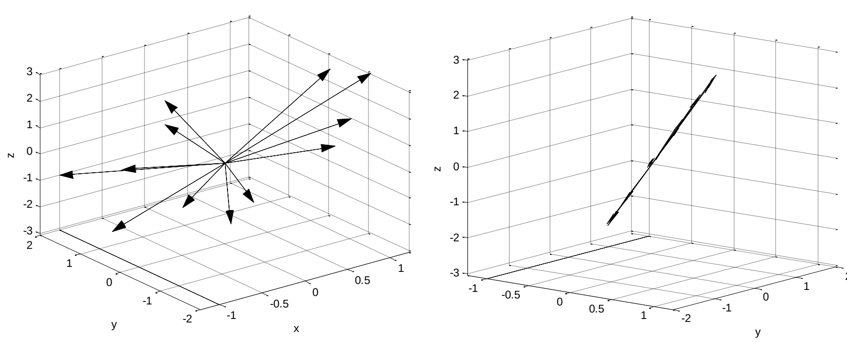
Example 6.26. In many applications, the standard choice for the orthonormal basis of a collection of (all possible real-valued) n-dimensional vectors is

$$\mathbf{e}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \mathbf{e}^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \mathbf{e}^{(n)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

$$\vec{v} = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 7 \\ 8 \\ 9 \end{pmatrix}$$

6.27. Suppose we start with a collection of M n-dimensional vectors. Do these M vectors really need to be represented in n dimensions?

Example 6.28. Figure 14a shows a particular collection of 10 vectors in 3-D. When viewed from appropriate angle (as in Figure 14b), we can see that they all reside on a 2-D plane. We only need a two-vector (orthonormal) basis. All ten vectors can be represented as linear combinations of these two vectors.



(a) (b)

Figure 14: Ten vectors on a plane

Example 6.29. Consider the four vectors below:

$$\mathbf{v}^{(1)} = \begin{pmatrix} -2 \\ -6 \\ 2 \end{pmatrix}, \mathbf{v}^{(2)} = \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}, \mathbf{v}^{(3)} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}, \text{ and } \mathbf{v}^{(4)} = \begin{pmatrix} 2 \\ 6 \\ -2 \end{pmatrix}.$$

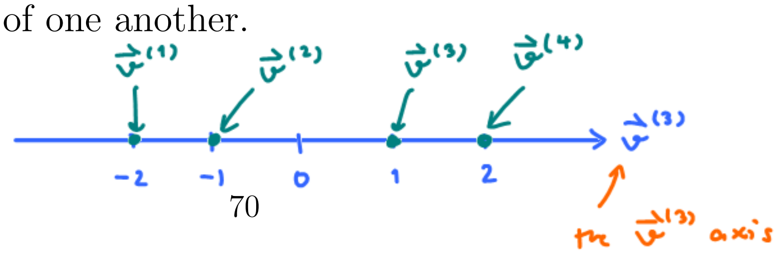
They are all multiples of one another.

$\vec{v}^{(1)} = (-2)\, \vec{v}^{(3)}$

$\vec{v}^{(2)} = (-1)\, \vec{v}^{(3)}$

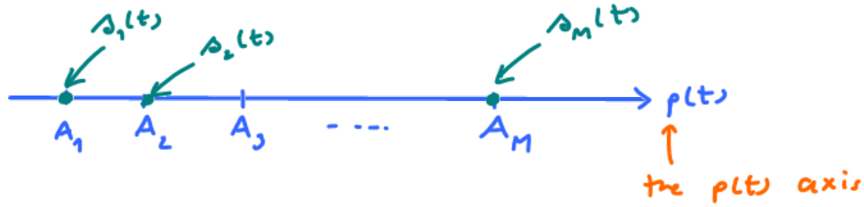
$\vec{v}^{(3)} = (1)\, \vec{v}^{(3)}$

$\vec{v}^{(4)} = (2)\, \vec{v}^{(3)}$



$s_1(t) = A_1\, p(t)$
$s_2(t) = A_2\, p(t)$
$\vdots$
$s_M(t) = A_M\, p(t)$

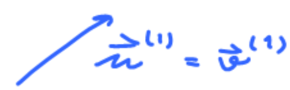
6.30. A sneak preview: Similar idea applies to waveforms. In PAM, we have M waveforms that are simply multiples of a pulse $p(t)$. Therefore, one may represent them in one dimension as



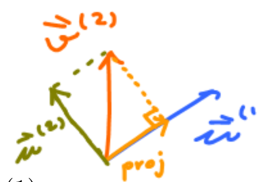
6.31. **Gram-Schmidt Orthogonalization Procedure (GSOP)** for constructing a collection of orthonormal vectors from a set of n-dimensional vectors $\mathbf{v}^{(i)}$, $1 \le i \le M$.

(a) Arbitrarily select a (nonzero) vector from the set, say, $\mathbf{v}^{(1)}$.
Let $\mathbf{u}^{(1)} = \mathbf{v}^{(1)}$.
Normalize its length to obtain the first vector, say,

$$\mathbf{e}^{(1)} = \frac{\mathbf{u}^{(1)}}{\left\|\mathbf{u}^{(1)}\right\|}.$$



(b) Select an unselected vector from the set, say, $\mathbf{v}^{(2)}$.
Subtract the projection of $\mathbf{v}^{(2)}$ onto $\mathbf{u}^{(1)}$:

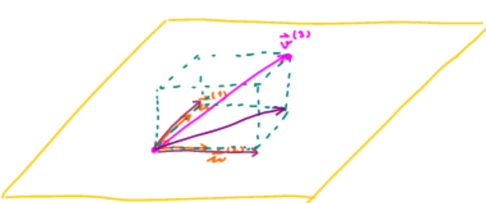


$$\mathbf{u}^{(2)} = \mathbf{v}^{(2)} - \operatorname{proj}_{\mathbf{u}^{(1)}}\left(\mathbf{v}^{(2)}\right) = \mathbf{v}^{(2)} - \frac{\left\langle \mathbf{v}^{(2)}, \mathbf{u}^{(1)} \right\rangle}{\left\langle \mathbf{u}^{(1)}, \mathbf{u}^{(1)} \right\rangle}\mathbf{u}^{(1)}$$
$$= \mathbf{v}^{(2)} - \left\langle \mathbf{v}^{(2)}, \mathbf{e}^{(1)} \right\rangle \mathbf{e}^{(1)}.$$

Then, we normalize the vector $\mathbf{u}^{(2)}$ to unit length:

$$\mathbf{e}^{(2)} = \frac{\mathbf{u}^{(2)}}{\left\|\mathbf{u}^{(2)}\right\|}.$$

(c) Continue by selecting an unselected vector from the set, say, $\mathbf{v}^{(3)}$ and subtract the projections of $\mathbf{v}^{(3)}$ into $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$:



$$\mathbf{u}^{(3)} = \mathbf{v}^{(3)} - \operatorname{proj}_{\mathbf{u}^{(1)}}\left(\mathbf{v}^{(3)}\right) - \operatorname{proj}_{\mathbf{u}^{(2)}}\left(\mathbf{v}^{(3)}\right)$$
$$= \mathbf{v}^{(3)} - \frac{\left\langle \mathbf{v}^{(3)}, \mathbf{u}^{(1)} \right\rangle}{\left\langle \mathbf{u}^{(1)}, \mathbf{u}^{(1)} \right\rangle}\mathbf{u}^{(1)} - \frac{\left\langle \mathbf{v}^{(3)}, \mathbf{u}^{(2)} \right\rangle}{\left\langle \mathbf{u}^{(2)}, \mathbf{u}^{(2)} \right\rangle}\mathbf{u}^{(2)}$$
$$= \mathbf{v}^{(3)} - \left\langle \mathbf{v}^{(3)}, \mathbf{e}^{(1)} \right\rangle \mathbf{e}^{(1)} - \left\langle \mathbf{v}^{(3)}, \mathbf{e}^{(2)} \right\rangle \mathbf{e}^{(2)}.$$

Then, we normalize the vector $\mathbf{u}^{(3)}$ to unit length:

$$\mathbf{e}^{(3)} = \frac{\mathbf{u}^{(3)}}{\left\|\mathbf{u}^{(3)}\right\|}.$$

(d) Continue this procedure for each of the remaining unselected vectors.

Example 6.32. Consider the four vectors in Example 6.24:

$$\mathbf{v}^{(1)} = \begin{pmatrix}1\\1\\0\end{pmatrix}, \mathbf{v}^{(2)} = \begin{pmatrix}1\\-1\\0\end{pmatrix}, \mathbf{v}^{(3)} = \begin{pmatrix}1\\1\\-1\end{pmatrix}, \text{ and } \mathbf{v}^{(4)} = \begin{pmatrix}-1\\-1\\-1\end{pmatrix}.$$

Use the Gram-Schmidt orthogonalization procedure (where the vectors are applied **in the order given**) to find the orthonormal vectors $\mathbf{e}^{(1)}, \mathbf{e}^{(2)}, \ldots$ that can be used to represent $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \mathbf{v}^{(3)}$, and $\mathbf{v}^{(4)}$.

$\vec{w}^{(1)} = \sqrt{2}\,\vec{e}^{(1)}$

$\vec{w}^{(1)} = \vec{v}^{(1)} = \begin{pmatrix}1\\1\\0\end{pmatrix}$ $\quad \|\vec{w}^{(1)}\| = \sqrt{2}$ $\quad \vec{e}^{(1)} = \frac{\vec{w}^{(1)}}{\|\vec{w}^{(1)}\|} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}$

$\vec{w}^{(2)} = \vec{v}^{(2)} - \underbrace{\text{proj}_{\vec{w}^{(1)}} \vec{v}^{(2)}} = \begin{pmatrix}1\\-1\\0\end{pmatrix}$ $\quad \|\vec{w}^{(2)}\| = \sqrt{2}$ $\quad \vec{e}^{(2)} = \frac{\vec{w}^{(2)}}{\|\vec{w}^{(2)}\|} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}$

$\vec{v}^{(2)} = \begin{pmatrix}1\\-1\\0\end{pmatrix}$

$\text{proj}_{\vec{w}^{(1)}} \vec{v}^{(2)} = \frac{\cancel{\langle \vec{v}^{(2)}, \vec{w}^{(1)} \rangle}^{\,0}}{\langle \vec{w}^{(1)}, \vec{w}^{(1)} \rangle} \vec{w}^{(1)} = 0\,\vec{w}^{(1)} = \vec{0}$ $\quad \vec{w}^{(2)} = \vec{v}^{(2)}$ $\quad \vec{w}^{(2)} = \sqrt{2}\,\vec{e}^{(2)}$

$\vec{w}^{(3)} = \vec{v}^{(3)} - \underbrace{\text{proj}_{\vec{w}^{(1)}} \vec{v}^{(3)}} - \underbrace{\text{proj}_{\vec{w}^{(2)}} \vec{v}^{(3)}} = \begin{pmatrix}0\\0\\-1\end{pmatrix}$

$\vec{v}^{(3)} = \begin{pmatrix}1\\1\\-1\end{pmatrix}$

$\text{proj}_{\vec{w}^{(1)}} \vec{v}^{(3)} = \frac{\langle \vec{v}^{(3)}, \vec{w}^{(1)} \rangle}{\langle \vec{w}^{(1)}, \vec{w}^{(1)} \rangle} \vec{w}^{(1)} = \frac{2}{2} \vec{w}^{(1)} = \vec{w}^{(1)} = \begin{pmatrix}1\\1\\0\end{pmatrix}$

$\text{proj}_{\vec{w}^{(2)}} \vec{v}^{(3)} = \frac{\langle \vec{v}^{(3)}, \vec{w}^{(2)} \rangle}{\langle \vec{w}^{(2)}, \vec{w}^{(2)} \rangle} \vec{w}^{(2)} = \frac{0}{\text{[illegible]}} \vec{w}^{(2)} = \vec{0}$

$\vec{w}^{(3)} = \vec{v}^{(3)} - \vec{w}^{(1)}$

$\vec{v}^{(3)} = \vec{w}^{(1)} + \vec{w}^{(3)}$

$\|\vec{w}^{(3)}\| = 1$

$\vec{e}^{(3)} = \vec{w}^{(3)}$

$\vec{e}^{(3)} = \frac{\vec{w}^{(3)}}{\|\vec{w}^{(3)}\|} = \vec{w}^{(3)} = \begin{pmatrix}0\\0\\-1\end{pmatrix}$

$\vec{0} = \vec{v}^{(4)} - (-1)\,\vec{w}^{(1)} - (1)\,\vec{w}^{(3)}$

$\vec{v}^{(4)} = (-1)\,\vec{w}^{(1)} + (1)\,\vec{w}^{(3)}$

$\cancel{\vec{w}^{(4)}} = \vec{v}^{(4)} - \underbrace{\text{proj}_{\vec{w}^{(1)}} \vec{v}^{(4)}}_{(-1)\,\vec{w}^{(1)}} - \underbrace{\text{proj}_{\vec{w}^{(2)}} \vec{v}^{(4)}}_{0\,\vec{w}^{(2)}} - \underbrace{\text{proj}_{\vec{w}^{(3)}} \vec{v}^{(4)}}_{(1)\,\vec{w}^{(3)}} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

discarded

Alternatively, we know that $n = 3$. We have found three orgothogonal vectors; so there can't be any more of them.

6.33. What did we get from GSOP?

(a) A collection of N orthogonal vectors $\mathbf{u}^{(1)}, \mathbf{u}^{(2)}, \ldots, \mathbf{u}^{(N)}$ where

$$N \le \min(M, n).$$

original # dimensions

vectors

(i) We discard the zero $u^{(k)}$ in the collection.

(ii) The $u^{(k)}$ are re-indexed to replace the skipped values.

This is then normalized to be a collection of N orthonormal vectors $\mathbf{e}^{(1)}, \mathbf{e}^{(2)}, \ldots, \mathbf{e}^{(N)}$.

(b) The collection $\mathbf{e}^{(1)}, \mathbf{e}^{(2)}, \ldots, \mathbf{e}^{(N)}$ forms an orthonormal basis for the span of $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \ldots, \mathbf{v}^{(M)}$.

Similarly, the collection $\mathbf{u}^{(1)}, \mathbf{u}^{(2)}, \ldots, \mathbf{u}^{(N)}$ forms an orthogonal basis for the span of $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \ldots, \mathbf{v}^{(M)}$.

(c) We can express $\mathbf{v}^{(j)}$ as

$$\mathbf{v}^{(j)} = \sum_{i=1}^{N} \mathrm{proj}_{\mathbf{e}^{(i)}}(\mathbf{v}^{(j)}) = \sum_{i=1}^{N} c_{i,j}\mathbf{e}^{(i)}$$

where $c_{i,j} = \left\langle \mathbf{v}^{(j)}, \mathbf{e}^{(i)} \right\rangle$. Then, the vector $\mathbf{c}^{(j)} = (c_{1,j}, c_{2,j}, \ldots, c_{N,j})^T$ gives the new coordinates of $\mathbf{v}^{(j)}$ based on the orthonormal basis from GSOP.

(d) In matrix form,

$$\vec{v}^{(j)} = \underbrace{\begin{bmatrix} \vec{e}^{(1)} & \vec{e}^{(2)} & \cdots & \vec{e}^{(N)} \end{bmatrix}}_{E} \underbrace{\begin{bmatrix} c_{1,j} \\ c_{2,j} \\ \vdots \\ c_{N,j} \end{bmatrix}}_{\vec{c}^{(j)} = \left[\vec{v}^{(j)}\right]_E}$$

Therefore, if we define $\mathbf{V} = \left[\mathbf{v}^{(1)}\ \mathbf{v}^{(2)}\ \cdots\ \mathbf{v}^{(M)}\right]$, we have

$$\underbrace{\begin{bmatrix} \vec{v}^{(1)} & \vec{v}^{(2)} & \cdots & \vec{v}^{(M)} \end{bmatrix}}_{V} = E \underbrace{\begin{bmatrix} \vec{c}^{(1)} & \vec{c}^{(2)} & \cdots & \vec{c}^{(M)} \end{bmatrix}}_{C}$$

So, we can look at the equation $\mathbf{V} = \mathbf{E}\mathbf{C}$ as a **decomposition** of the matrix $\mathbf{V}$. Because the vectors in $\mathbf{E}$ are orthonormal, we have $\mathbf{E}^H\mathbf{E} = \mathbf{I}$.

Example 6.34. Use the orthonormal vectors from the Examples 6.32 to construct the matrix $\mathbf{E} = \left[\mathbf{e}^{(1)}\ \mathbf{e}^{(2)}\ \cdots\right]$. Find the (upper triangular) matrix $\mathbf{C}$ such that $\mathbf{V} = \mathbf{E}\mathbf{C}$ where $\mathbf{V} = \left[\mathbf{v}^{(1)}\ \mathbf{v}^{(2)}\ \mathbf{v}^{(3)}\ \mathbf{v}^{(4)}\right]$.

$$\vec{v}^{(1)} = \begin{pmatrix}1\\1\\0\end{pmatrix} = 1\vec{w}^{(1)} + 0\vec{w}^{(2)} + 0\vec{w}^{(3)} = \sqrt{2}\,\vec{e}^{(1)} + 0\,\vec{e}^{(2)} + 0\,\vec{e}^{(3)} = \overbrace{\left[\vec{e}^{(1)}\ \vec{e}^{(2)}\ \vec{e}^{(3)}\right]}^{E}\overbrace{\begin{pmatrix}\sqrt{2}\\0\\0\end{pmatrix}}^{\vec{c}^{(1)}}$$

$$\vec{v}^{(2)} = \begin{pmatrix}1\\-1\\0\end{pmatrix} = 0\vec{w}^{(1)} + 1\vec{w}^{(2)} + 0\vec{w}^{(3)} = 0\,\vec{e}^{(1)} + \sqrt{2}\,\vec{e}^{(2)} + 0\,\vec{e}^{(3)} = E\begin{pmatrix}0\\\sqrt{2}\\0\end{pmatrix} = E\,\vec{c}^{(2)}$$

$$\vec{v}^{(3)} = \begin{pmatrix}1\\1\\-1\end{pmatrix} = 1\vec{w}^{(1)} + 0\vec{w}^{(2)} + 1\vec{w}^{(3)} = \sqrt{2}\,\vec{e}^{(1)} + 0\,\vec{e}^{(2)} + 1\,\vec{e}^{(3)} = E\begin{pmatrix}\sqrt{2}\\0\\1\end{pmatrix} = E\,\vec{c}^{(3)}$$

$$\vec{v}^{(4)} = \begin{pmatrix}-1\\-1\\-1\end{pmatrix} = (-1)\vec{w}^{(1)} + 0\vec{w}^{(2)} + 1\vec{w}^{(3)} = (-\sqrt{2})\,\vec{e}^{(1)} + 0\,\vec{e}^{(2)} + 1\,\vec{e}^{(3)} = E\begin{pmatrix}-\sqrt{2}\\0\\1\end{pmatrix} = E\,\vec{c}^{(4)}$$

$$\underbrace{\left[\vec{v}^{(1)}\ \vec{v}^{(2)}\ \vec{v}^{(3)}\ \vec{v}^{(4)}\right]}_{V} = E\underbrace{\left[\vec{c}^{(1)}\ \vec{c}^{(2)}\ \vec{c}^{(3)}\ \vec{c}^{(4)}\right]}_{C}$$

upper triangular

$$V = EC \qquad C = \begin{bmatrix}\sqrt{2} & 0 & \sqrt{2} & -\sqrt{2}\\ 0 & \sqrt{2} & 0 & 0\\ 0 & 0 & 1 & 1\end{bmatrix}$$

6.35. Important properties: the transformation from $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \ldots, \mathbf{v}^{(M)}$ to $\mathbf{c}^{(1)}, \mathbf{c}^{(2)}, \ldots, \mathbf{c}^{(M)}$ preserve many geometric quantities.

Parseval's identity → (a) Same inner product.

$$\langle \vec{v}^{(i)}, \vec{v}^{(j)}\rangle = \left(\vec{v}^{(j)}\right)^H \vec{v}^{(i)} = \left(E\vec{c}^{(j)}\right)^H\left(E\vec{c}^{(i)}\right) = \left(\vec{c}^{(j)}\right)^H \underbrace{\cancel{E^H E}}_{I}\,\vec{c}^{(i)} = \left(\vec{c}^{(j)}\right)^H \vec{c}^{(i)} = \langle \vec{c}^{(i)}, \vec{c}^{(j)}\rangle$$

(b) Same norm.

$$\|\vec{v}^{(j)}\| = \sqrt{\langle \vec{v}^{(j)}, \vec{v}^{(j)}\rangle} = \sqrt{\langle \vec{c}^{(j)}, \vec{c}^{(j)}\rangle} = \|\vec{c}^{(j)}\|$$

(c) Same distance.

$$d\left(\vec{v}^{(i)}, \vec{v}^{(j)}\right) = \|\vec{v}^{(j)} - \vec{v}^{(i)}\| = \|\vec{c}^{(j)} - \vec{c}^{(i)}\| = d\left(\vec{c}^{(i)}, \vec{c}^{(j)}\right)$$

$$d\left(\vec{v}^{(i)}, \vec{v}^{(j)}\right) = \|\vec{v}^{(j)} - \vec{v}^{(i)}\|$$